Last Time: Predicates, Quantifiers, Sets.

Today: More...

Recall: A predicate is a statement w/ variables.

The variables represent elements of a domain set. ("subjects")

Ex: $L(x,y)$: "x loves y"

Domain: $\{z : z \text{ is a person}\}$

A quantifier tells us how many subjects satisfy a predicate statement (roughly...)

Universal Quantifier: $\forall x\, P(x)$ = "for all x, $P(x)$".

Existential Quantifier: $\exists x\, P(x)$ = "At least one x satisfies $P(x)$"

i.e. "There exists an x satisfying $P(x)$"

Uniqueness Quantifier: $\exists! (x \in \mathbb{D})\, P(x)$ = "There is a unique x satisfying $P(x)$"

Unique means "exactly one"

So "$\exists!$" reads "there is one and only one".

↕ The Highlander Quantifier...

There can be only one...

Ex: Translate the statement:

"There is a unique natural number, n, so that for all natural numbers, m, $n+m=m$"

Sol: We rewrite

"There exists a unique $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have $n+m=m$"

- $n \in \mathbb{N}$: set of natural numbers
- "There exists a unique": quantifier
- such that: "scoping"
- For all: quantifier
- we have: "scoping"
- $n+m=m$: predicate

Now this becomes: $\exists! n \in \mathbb{N}\left[\forall m \in \mathbb{N}[n+m=m]\right]$. ☑

Q: How does negation interact w/ Quantification?

A: Note that $\forall x \in D\,[P(x)]$ means "every x in D satisfies $P(x)$". If $D = \{d_1, d_2, \cdots, d_n\}$ then $\forall x \in D\,[P(x)]$ is equivalent to $P(d_1) \wedge P(d_2) \wedge \cdots \wedge P(d_n)$.

So $\neg \forall x \in D\,[P(x)]$ would mean $(\neg P(d_1)) \vee (\neg P(d_2)) \vee \cdots \vee (\neg P(d_n))$

OTOH $\exists x \in D\,[Q(x)]$ would mean (for finite D):

$Q(d_1) \vee Q(d_2) \vee \cdots \vee Q(d_n)$

Point: $\forall$ ~ "big and" $\exists$ ~ "big or"

So: $\neg \forall_x P(x) \equiv \exists_x [\neg P(x)]$

and $\neg \exists_x P(x) \equiv \forall_x [\neg P(x)]$

← De Morgan's Quantifier Laws

Ex: Negate $\exists_{x \in \mathbb{N}} \forall_{y \in \mathbb{N}} [x+y = y]$.

Sol: $\neg \exists_{x \in \mathbb{N}} \forall_{y \in \mathbb{N}} [x+y = y]$

$\equiv \forall_{x \in \mathbb{N}} [\neg \forall_{y \in \mathbb{N}} [x+y = y]]$ (De Morgan)

$\equiv \forall_{x \in \mathbb{N}} \exists_{y \in \mathbb{N}} [\neg (x+y = y)]$ ▨

Set Theory

A set is a collection of elements (i.e. objects in the set).

We write $x \in S$ to mean "x is an element of S".

Two sets are equal when they have the same elements.

(So: order and repeated elements don't change the set...)

Notations: between the braces ...

Roster notation
List out all elements

Ex: $\{1, 2, 3, 5, 12, -15\}$

Set-Builder
Describe the elements

Ex: $\{x \in \mathbb{N} : x \text{ is even}\}$.

Logically equivalent predicates give the same set...

Set Operations

Goal: Build new sets from old...

Running Example: Let $A = \{1, 2, -3, 5, -7\}$,
$B = \{-1, 4, 5\}$, $C = \{1, 5, -7\}$.

Def^n: The cardinality of set S is the number of elements it contains. We denote the cardinality of S by $|S|$ or $\#S$.

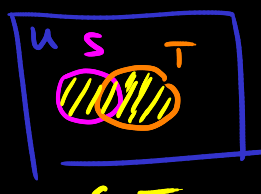
Ex: $|A| = |\{1, 2, -3, 5, -7\}| = 5 = \#A$,

$|B| = |\{-1, 4, 5\}| = 3$, $|C| = |\{1, 5, -7\}| = 3$

"equicardinal" i.e. have same cardinality.

Def^n: The union of sets S and T is

$$S \cup T := \{x : x \in S \text{ or } x \in T\}.$$



Ex: $A \cup B = \{1, 2, -3, 5, -7\} \cup \{-1, 4, 5\}$

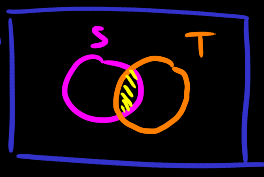
$= \{1, 2, -3, 5, -7, -1, 4, 5\}$

$= \{1, 2, -3, 5, -7, -1, 4\}$

$B \cup C = \{-1, 4, 5\} \cup \{1, 5, -7\} = \{-1, 4, 5, 1, -7\}$

Def: The intersection of sets S and T is

$$S \cap T = \{x : x \in S \text{ and } x \in T\}$$



Ex: $A \cap B = \{1, 2, -3, 5, -7\} \cap \{-1, 4, 5\} = \{5\}$

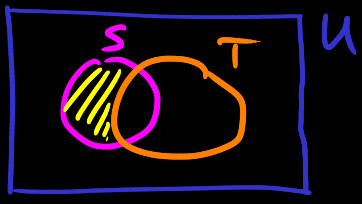
"singleton" b/c it has cardinality 1.

Ex: $\{1, 3, 7\} \cap \{2, 4, 8\} = \{\} = \emptyset$

"empty set" or "null set"

Def^n: The set difference, S remove T is

$$S \setminus T = \{x : x \in S \text{ but } x \notin T\}$$



Ex: $A \setminus B = \{1, 2, -3, 5, -7\} \setminus \{-1, 4, 5\}$

$= \{1, 2, -3, -7\}$

$C \setminus B = \{1, 5, -7\} \setminus \{-1, 4, 5\} = \{1, -7\}$

Def^n: The Cartesian product of set S with T is

$$S \times T = \{(x, y) : x \in S \text{ and } y \in T\}$$

Ex: Order Matters for cartesian product!!! $B \times C \neq C \times B$.

$B \times C = \{-1, 4, 5\} \times \{1, 5, -7\}$

$= \{(-1, 1), (-1, 5), (-1, -7),$
$(4, 1), (4, 5), (4, -7),$
$(5, 1), (5, 5), (5, -7)\}$

$C \times B = \{1, 5, -7\} \times \{-1, 4, 5\}$

$= \{(1, -1), (1, 4), (1, 5),$
$(5, -1), (5, 4), (5, 5),$
$(-7, -1), (-7, 4), (-7, 5)\}$

Defn: The power set of set S is

$$\mathcal{P}(S) = \{ x : x \subseteq S \}$$

Recall: we write $X \subseteq Y$ for the statement $\forall x \in X [x \in Y]$

i.e. $\forall x [x \in X \to x \in Y]$.

Ex: $\mathcal{P}(C) = \{ X : X \subseteq C \}$ $\quad C = \{1, 5, -7\}$

$$= \{ \{\}, \{1\}, \{5\}, \{-7\}, \{1,5\}, \{1,-7\}, \{5,-7\}, \{1,5,-7\} \}$$

card 0 | card 1 | card 2 | card 3

$\mathcal{P}(\emptyset) = \mathcal{P}(\{\}) = \{\{\}\}$. **Note:** $|\mathcal{P}(\emptyset)| = 1$

card 1

card 0

$= \{\emptyset\}$

Next Time: Proofs involving Sets!